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STRESS SINGULARITIES AT VERTICES OF CONICAL INCLUSIONS WITH FREELY SLIDING INTERFACES

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Abstract—The axisymmetric problem of the elastic stress singularity at the vertex of a conical inclusion bonded into a conical notch is analyzed. The shear stress is assumed to vanish along the interface while the normal stress is fully transmitted. This corresponds to the low load–low deformation regime of high temperature materials, when the grain boundaries and the interfaces of second phase particles may slide viscously. The stress and displacement fields are expressed in terms of spherical harmonic functions and the singularity exponent is obtained from the solution of the eigenproblem defined by the boundary conditions. Supersingularities of the stress field close to the vertex were found for certain configurations. In all analyzed cases, only real singularities were obtained. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The nucleation of cracks and cavities in engineering polycrystals occurs under the enhanced local stress field around concentrators such as second phase inclusions and grain triple junctions. In high temperature alloys, loaded at temperatures higher than half the melting point T_m, the stress is further concentrated due to the sliding of the grain boundaries. For pseudo-brittle materials working under severe high temperature conditions and for thin films in which the dislocation activity is inhibited while the grain boundaries slide viscously, a precise characterization of the internal stress state which may induce microcracking becomes important.

When the grain boundaries are not allowed to slide, the elastic stress field becomes singular at grain triple junctions due to the thermal and elastic anisotropy of the grains. It was shown (Chiu, 1977; Evans, 1978) that the thermal anisotropy induces a weak logarithmic-type singularity while the elastic anisotropy leads to a power-law function for the elastic stress fields close to the triple junction (Tvergaard and Hutchinson, 1988). This problem was extensively studied in both two dimensions (Tvergaard and Hutchinson, 1988; Picu and Gupta, 1994) and three dimensions (Somaratna and Ting, 1986; Ghahremani *et al.*, 1990). As discussed by Ghahremani *et al.* (1990), the singularity exponents are significantly stronger at a 3D vertex compared to those at an equivalent 2D apex.

Local stress concentration occurs at triple junctions and corners in the grain boundaries in two-phase polycrystals even in the absence of anisotropy, due to the difference in the elastic constants of the two phases. The related 2D problem of an infinite composite plane was analyzed by Bogy and Wang (1971) who found the singularity exponent of the stress field around a corner in the interface between two materials for a variety of geometrical configurations and material parameters. The 3D problem of stress concentration at vertices of inclusions of a second phase was analyzed in the axisymmetric case by Keer and Parihar (1978) who extended an earlier study by Bazant and Keer (1974). These studies suggest that the concentration effect is weaker in plane than in 3D. Moreover, the singularities obtained by Bogy and Wang (1971) for the composite plane are weaker than -0.5, as in a standard crack problem. Finally, when comparing the effect of the anisotropy with that due to the presence of a second phase inclusion, for realistic levels of anisotropy and material constants, the inclusion case is often more critical.

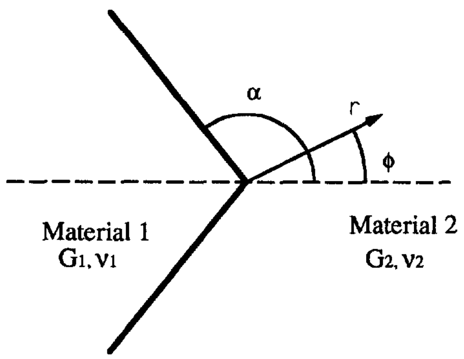

Fig. 1. Details at a conical vertex.

When the grain boundaries are allowed to slide, the stresses are further concentrated at triple junctions and at grain boundary pinning particles. The problem was analyzed in 2D by Lau *et al.* (1983, 1984) by prescribing a linear viscous behavior for the sliding grain boundaries and considering the grains to flow according to a power-law function. The largest singularity obtained for an alloy with an effective creep exponent of 3 was of -0.225 at the apex of a square pinning particle and of -0.231 at a triple junction with grain boundaries oriented 120° apart. When the polycrystal is loaded at higher strain rates, the grains behave essentially within their elastic limit while the grain boundaries are still sliding in a viscous way. At a given strain rate, there will be an intrinsic shear resistance a boundary opposes to sliding. Once this resistance is overcome by the applied shear stress, the boundary is free to slide. Such behavior was documented (Picu and Gupta, 1995a) to occur in polycrystalline ice at high homologous temperatures ($>0.9\ T_m$) when loading at strain rates as high as $10^{-2}\ \mathrm{s}^{-1}$. In these experiments grain boundary sliding-induced cracks nucleated from the triple junctions or from acute corners on the grain boundaries. Thus, it becomes important to determine the elastic stress field around a triple junction when the boundaries are free to slide.

The associated 2D elasticity problem of grain boundary sliding-induced stress concentration at triple junctions was analyzed by Picu and Gupta (1995b). In their study, the three grains forming the triple junction were considered to be made from the same isotropic elastic material (single-phase system) and the grain boundaries were allowed to slide freely while fully transmitting the normal stresses. The singularity exponents result to be independent of the elastic constants of the grains and are only functions of the geometric parameters. For several configurations of the triple junction, supersingularities (i.e. singularities stronger than -0.5) of the stress field were found.

The purpose of the present note is to analyze the effect of the shear stress relaxation along the interface of a conical inclusion, on the stress field at its vertex. The two materials of the inclusion and of the matrix are considered to be isotropic and to behave elasticaly. The interface transmits the normal stresses while is allowed to slide freely. This extends the study in (Picu and Gupta, 1995b) to the three-dimensional axisymmetric case.

2. ANALYSIS

Consider the geometry of Fig. 1. The two materials are defined by their shear moduli G_1 and G_2 and the Poisson's ratios ν_1 and ν_2. In a spherical coordinate system (r, θ, ϕ) centered at the vertex, the position of the interface is defined by the angle $\phi = \alpha$. The stress field close to the vertex can be written in a standard separable form as

$$\sigma_{ij}(r, \theta, \phi) \propto r^{\lambda-1} f(\theta, \phi), \tag{1}$$

where $(\lambda - 1)$ is the singularity exponent and f is a function which depends only on ϕ and θ.

The problem of equilibrium of a symmetrically loaded body of revolution can be subdivided into a problem of torsion and one of deformation in the meridional plane. Since the torsion is not of interest here, only the second problem is considered. In this case, the displacement components are independent of θ and the equilibrium equations in displacements lead to a Legendre-type differential equation (Lure, 1964), the solutions of which are the generalized Legendre functions $P_\lambda(\cos\phi)$. Following Thompson and Little (1970), the displacement field in each region can be expressed as

$$u_r(r,\phi) = r^\lambda\left[a\left(1-\frac{\lambda+1}{4(1-\nu)}\right)\cos\phi P_\lambda(\cos\phi)+b(1+\lambda)P_{\lambda+1}(\cos\phi)\right]$$

$$u_\phi(r,\phi) = -r^\lambda\left\{\left(b-\frac{a}{4(1-\nu)}\right)\frac{\partial P_{\lambda+1}(\cos\phi)}{\partial(\cos\phi)}+a\left[1+\frac{\lambda}{4(1-\nu)}\right]P_\lambda(\cos\phi)\right\}\sin\phi \quad (2)$$

where a and b are constants to be determined from the solution of the full boundary value problem. The stresses σ_{rr}, $\sigma_{\phi\phi}$ and $\sigma_{r\phi}$ can be readily derived from (2) and are given in (Thompson and Little, 1970).

The boundary conditions of the elasticity problem at hand are

$$\begin{aligned}\sigma^1_{\phi\phi}(r,\pi-\alpha) &= \sigma^2_{\phi\phi}(r,\alpha)\\ u^1_\phi(r,\pi-\alpha) &= u^2_\phi(r,\alpha)\\ \sigma^1_{r\phi}(r,\pi-\alpha) &= 0\\ \sigma^2_{r\phi}(r,\alpha) &= 0\end{aligned} \quad (3)$$

where the last two equations represent the free sliding condition along the interface at $\phi = \alpha$, and the indices 1 and 2 refer to the two grains. In the local field analysis, λ is obtained as the eigenvalue of the 4×4 system of equations that result after the use of the boundary conditions (3). The exponent λ is, in general, a complex-valued quantity with $\mathrm{Re}(\lambda)-1$ representing the power of the singularity, and $\mathrm{Im}(\lambda)$ leading to oscillations in the fields close to the vertex. The domain of interest for the variation of $\mathrm{Re}(\lambda)$ is $0 < \mathrm{Re}(\lambda) < 1$, since if $\mathrm{Re}(\lambda) > 1$ the fields are nonsingular and if $\mathrm{Re}(\lambda) < 0$, the strain energy density becomes unbounded. Therefore, the singularity exponent was searched numerically in the domain $(0 < \mathrm{Re}(\lambda) < 1;\ 0 \leqslant \mathrm{Im}(\lambda) \leqslant 1)$ of the complex plane using an algorithm similar with the one presented by Somaratna and Ting (1986).

3. RESULTS AND CONCLUSIONS

Let us consider first the case when the matrix and the conical inclusion are made out of the same material ($G_1 = G_2$, $\nu_1 = \nu_2 = \nu$). This is similar with considering the shear stress to vanish at all points of a conical surface inside an homogeneous elastic solid. As shown before, the stresses are singular close to the vertex and the singularity exponent depends upon the cone angle α and the Poisson's ratio ν. Figure 2 shows the variation of the singularity $(\lambda-1)$ with the angle α, for several values of ν. Here and in the following, only real singularity exponents were found such that the solution is free from oscillations of the stress and displacement fields close to the vertex. For angles α larger than 160° and for $\nu > 0.4$, supersingularities were obtained. If $\alpha = 180°$, the continuity of normal displacements and stresses is directly satisfied and the problem becomes indeterminate. Moreover, if $\alpha = 90°$, the two half spaces are free to slide one over another and the global equilibrium is no longer satisfied. Hence, the solution was not extended toward the limits of the interval [90°, 180°].

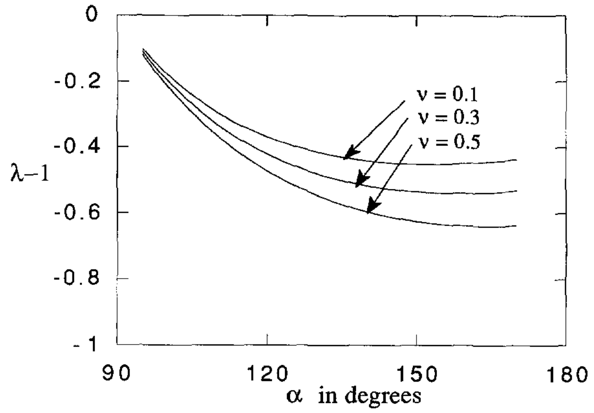

Fig. 2. Variation of the singularity exponent $(\lambda-1)$ with the angle α when considering the same material on both sides of the conical interface. The solution depends only upon the Poisson's ratio ν and the angle α.

Figure 3 shows the variation of the exponent $(\lambda-1)$ with α, for $G_1/G_2 = 10$ and $\nu_1 = \nu_2 = \nu$. The broken line (a) (Keer and Parihar, 1978) corresponds to the case when the interface between the two materials fully transmits the shear and normal stresses. For this curve, $G_1/G_2 = 10$ and $\nu_1 = \nu_2 = 0.3$. Hence, it represents only the effect of the elastic

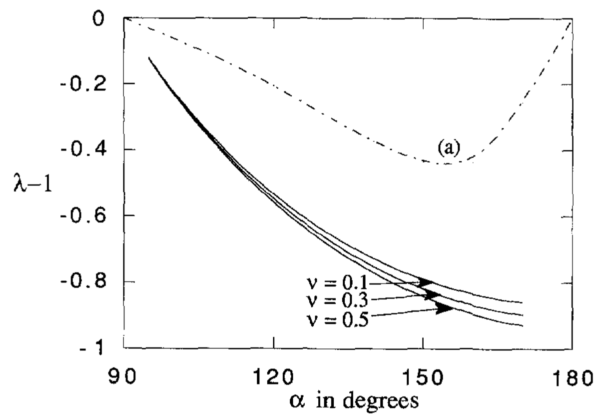

Fig. 3. Variation of the singularity exponent $(\lambda-1)$ with the cone angle α for the case when $G_1/G_2 = 10$ and $\nu_1 = \nu_2 = \nu$. Curve (a) (Keer and Parihar, 1978) represents the variation of $(\lambda-1)$ with α for a fully bounded interface between two materials with $G_1/G_2 = 10$ and $\nu_1 = \nu_2 = 0.3$.

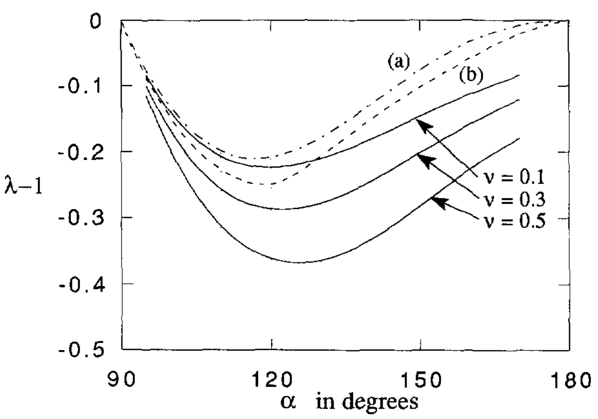

Fig. 4. Variation of the singularity exponent $(\lambda-1)$ with the cone angle α for the case when $G_1/G_2 = 0.1$ and $\nu_1 = \nu_2 = \nu$. Curve (a) (Keer and Parihar, 1978) corresponds to a fully bounded interface between two materials with $G_1/G_2 = 0.1$ and $\nu_1 = \nu_2 = 0.3$, while (b) (Bazant and Keer, 1974) corresponds to a conical notch of angle $\pi-\alpha$ in a material with $\nu = 0.3$.

mismatch on the stress field singularity. When the shear stress along the interface vanishes, the two effects add, and a significant increase in the power of the singularity results. This may be important at high homologous temperatures and high strain rates when the stress field at the vertex can readily become supersingular. Since in this case the singularity exponent is stronger than the crack-like one of -0.5, crack nucleation from the vertex is energetically favorable under the short range singular stress field there.

In Fig. 4, the same variation of $(\lambda-1)$ with α when $G_1/G_2 = 0.1$ and $\nu_1 = \nu_2 = \nu$ is shown. The broken line (a) (Keer and Parihar, 1978) corresponds to a fully bounded interface between two materials with $G_1/G_2 = 0.1$ and $\nu_1 = \nu_2 = 0.3$, while (b) (Bazant and Keer, 1974) represents the variation of the singularity at the vertex of a conical notch of angle $\pi-\alpha$ in a material with $\nu = 0.3$. In this case, the effect of shear stress relaxation is smaller than before and eventually decreases when the elastic constants of material 1 tend to zero. Hence, the most critical configurations are the ones with $G_1/G_2 > 1$ when, after the relaxation of the shear stress along the interface, crack nucleation conditions form the vertex are met, as discussed before.

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